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非線形固有値問題の解の漸近挙動

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1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$(1.1) \quad -u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0, 1),$$

$$(1.2) \quad u(t) > 0, \quad t \in I,$$

$$(1.3) \quad u(0) = u(1) = 0,$$

where $\lambda > 0$ is an eigenvalue parameter. We assume that $f(u)$ satisfies the following conditions (A.1)–(A.3).

(A.1) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $g(u) := f(u)/u$ is strictly increasing for $u \geq 0$ ($g(0) := 0$).

(A.3) $g(u) \rightarrow \infty$ as $u \rightarrow \infty$.

The typical examples of $f(u)$ which satisfy (A.1)–(A.3) are

$$f(u) = u^p, \quad f(u) = u^{p+2}/(1 + u^2), \quad f(u) = u^p e^u \quad (p > 1).$$

We know from [1] that for each given $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ with $\|u_\alpha\|_2 = \alpha$. Furthermore, The set $\{(\lambda(\alpha), u_\alpha) : \alpha > 0\}$ gives all solutions and is an unbounded curve of class C^1 in $\mathbf{R}_+ \times L^2(I)$ emanating from $(\pi^2, 0)$.

The purpose here is to study precisely the global structure of this bifurcation branch in $\mathbf{R}_+ \times L^2(I)$. To do this, we establish several types of precise asymptotic formulas for $\lambda(\alpha)$ as $\alpha \rightarrow \infty$ under some additional conditions on f .

We know from [1] that for $t \in \bar{I}$,

$$(1.4) \quad g^{-1}(\lambda - \pi^2) \sin \pi t \leq u_\lambda(t) \leq g^{-1}(\lambda).$$

In particular, put $t = \frac{1}{2}$. Then as $\lambda \rightarrow \infty$

$$(1.5) \quad g^{-1}(\lambda - \pi^2) \leq \|u_\lambda\|_\infty \leq g^{-1}(\lambda).$$

Therefore, for $\lambda \gg 1$,

$$(1.6) \quad \lambda = g(\|u_\lambda\|_\infty) + O(1).$$

For instance, let $f(u) = u^p$. Then since $g(u) = f(u)/u = u^{p-1}$, for $\lambda \gg 1$

$$(1.7) \quad \lambda = \|u_\lambda\|_\infty^{p-1} + O(1).$$

Furthermore, we know that as $\lambda \rightarrow \infty$

$$(1.8) \quad \frac{u_\lambda(t)}{g^{-1}(\lambda)} \rightarrow 1$$

uniformly on any compact set in I . Then we obtain

$$\alpha = \|u_\alpha\|_2 = \left(\int_I g^{-1}(\lambda)^2 dt \right)^{1/2} (1 + o(1)) = g^{-1}(\lambda)(1 + o(1)).$$

This implies that, in many cases,

$$(1.9) \quad \lambda(\alpha) = g(\alpha) + o(g(\alpha)).$$

For instance, let $f(u) = u^p$. Then for $\alpha \gg 1$,

$$(1.10) \quad \lambda(\alpha) = \alpha^{p-1} + o(\alpha^{p-1}).$$

This asymptotic formula has been improved as follows.

Theorem 1 [6]. *Let $f(u) = u^p$ ($p > 1$). Further, let an arbitrary $n \in \mathbf{N}_0$ be fixed. Then as $\alpha \rightarrow \infty$*

$$\lambda(\alpha) = \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + \sum_{k=0}^n \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}),$$

where

$$C_1 = (p+3) \int_I \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1} s^{p+1}} ds$$

and $a_k(p)$ ($\deg a_k(p) \leq k+1$) is the polynomial determined inductively by a_0, a_1, \dots, a_{k-1} .

For instance, we have

$$a_0(p) = 1, \quad a_1(p) = \frac{(5-p)(9-p)}{24}, \quad a_2(p) = \frac{(3-p)(5-p)(7-p)}{24}.$$

We also obtain the information about the slope of the boundary layer of u_α for $\alpha \gg 1$.

Theorem 2 [6]. Let $f(u) = u^p$ ($p > 1$). Further, let an arbitrary $n \in \mathbf{N}_0$ be fixed. Then as $\alpha \rightarrow \infty$

$$\begin{aligned} u'_\alpha(0)^2 &= u'_\alpha(1)^2 = \frac{p-1}{p+1} \alpha^{p+1} + C_1 \alpha^{(p+3)/2} + \sum_{k=0}^n \frac{2A_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{2+k(1-p)/2} \\ &\quad + o(\alpha^{2+n(1-p)/2}), \end{aligned}$$

where $A_k(p)$ ($\deg A_k(p) \leq k+1$) is the polynomial determined by a_0, a_1, \dots, a_{k-1} .

For instance,

$$A_0(p) = 1, \quad A_1(p) = \frac{(9-p)(13-p)}{48}, \quad A_2(p) = \frac{(5-p)(7-p)(9-p)}{48}.$$

So it is natural to consider the following problem. Consider $f(u)$ which satisfies (A.1)–(A.3).

Then is the following formula valid or not for $\alpha \gg 1$?

$$(1.11) \quad \lambda(\alpha) = g(\alpha) + B_1 g(\alpha)^{1/2} + \dots,$$

where B_1 is a constant. To treat this problem, we assume additional conditions. Let $f(u) = u^p h(u)$ ($p > 1$). Assume that $h(u)$ is C^2 function for $u \geq 0$. Besides, $h(u)$ satisfies the following conditions (B.1)–(B.4).

(B.1) As $u \rightarrow \infty$

$$(1.12) \quad \frac{uh'(u)}{h(u)} \rightarrow 0.$$

Furthermore, there exists a constant $C_0 \geq 0$ such that as $u \rightarrow \infty$

$$(1.13) \quad uh'(u) \rightarrow C_0.$$

(B.2) There exist constants $C > 0$ and $\delta > 0$ such that for $u \gg 1$

$$(1.14) \quad |h'(u) + uh''(u)| \leq Cu^{-(1+\delta)}.$$

(B.3) For $0 \leq a \leq 1$ and $u \gg 1$

$$(1.15) \quad \frac{h(au)}{h(u)} \leq C.$$

Furthermore, for a fixed $0 < a \leq 1$, as $u \rightarrow \infty$

$$(1.16) \quad \frac{h(au)}{h(u)} \rightarrow 1.$$

(B.4) (a) $u^{p+1}|h'(u)|$ is non-decreasing for $u \geq 0$ or,

(b) $u^{p+1}|h'(u)|$ is bounded for $u \geq 0$.

The typical examples of h are: (i) $h(u) \equiv 1$, (ii) $h(u) = \log(u+1)$, (iii) $h(u) = u^2/(1+u^2)$.

Theorem 3 [7]. *Let $p > 1$ be fixed. Assume that $f(u) := u^p h(u)$ satisfies (A.1)–(A.3) and (B.1)–(B.4). Then as $\alpha \rightarrow \infty$*

$$\lambda(\alpha) = \alpha^{p-1}h(\alpha) + \frac{1}{p+1}C_0\alpha^{p-1} + (p+3)C_1\alpha^{(p-1)/2}\sqrt{h(\alpha)}(1+o(1)).$$

Remark 4. (i) For $\alpha \gg 1$, by (B.2), we see that

$$(1.17) \quad C_0 = \alpha h'(\alpha)(1+o(1)).$$

Therefore, as $\alpha \rightarrow \infty$

$$(1.18) \quad \frac{\alpha^{p-1}C_0}{\alpha^{p-1}h(\alpha)} = \frac{\alpha^p h'(\alpha)(1+o(1))}{\alpha^{p-1}h(\alpha)} \rightarrow 0.$$

So we find that the leading term of $\lambda(\alpha)$ in Theorem 3 is $\alpha^{p-1}h(\alpha)$.

(ii) If $C_0 \neq 0$, then the second term of $\lambda(\alpha)$ is $C_0\alpha^{p-1}/(p+1)$. Therefore, our conjecture (1.11) is valid if and only if $C_0 = 0$. We note that, if $h(u) = \log(u+1)$, then $C_0 = 1$. Further, if $h(u) = u^2/(1+u^2)$, then $C_0 = 0$.

Now we consider the case where $f(u) = u^p e^u$ ($p > 1$).

Theorem 5 [8]. *Assume that $f(u) = u^p e^u$ ($p > 1$) in (1.1). Then as $\alpha \rightarrow \infty$*

$$\lambda(\alpha) = \alpha^{p-1}e^\alpha + \frac{\pi}{4}\alpha^{(p+1)/2}e^{\alpha/2}(1+o(1)).$$

2 Sketch of the proof of Theorem 3

We begin with notations and the fundamental properties of $\lambda(\alpha)$ and u_α . Let $F(u) := \int_0^u f(s)ds$. Let $\|\cdot\|_q$ ($1 \leq q \leq \infty$) denote the usual L^q -norm. C denotes various positive constants independent of $\alpha \gg 1$. It is known by [1] that (1.1)–(1.3) has a unique solution u_α for a given $\alpha > 0$ and the mapping $\alpha \mapsto u_\alpha \in C^2(\bar{I})$ is C^1 for $\alpha > 0$. By (1.4) and (1.5), for $\alpha \gg 1$

$$(2.1) \quad \lambda(\alpha) = \alpha^{p-1}h(\alpha) + o(\alpha^{p-1}h(\alpha)),$$

$$(2.2) \quad u_\alpha(t) = \|u_\alpha\|_\infty(1 + o(1)) = \alpha(1 + o(1)), \quad t \in I.$$

We put

$$(2.3) \quad \lambda_1(\alpha) := \lambda(\alpha) - \alpha^{p-1}h(\alpha),$$

$$(2.4) \quad \gamma(\alpha) := \|u'_\alpha\|_2^2 + 2 \int_I F(u_\alpha(t))dt.$$

To show Theorem 3, we find $\lambda_1(\alpha)$ when $\alpha \gg 1$. To do this, we define the second term $\gamma_1(\alpha)$ of $\gamma(\alpha)$, which plays important roles, as follows.

$$(2.5) \quad \gamma_1(\alpha) := \gamma(\alpha) - \frac{2}{p+1}\alpha^{p+1}h(\alpha).$$

The rough idea of the proof is as follows.

- (i) We obtain three estimates in Lemmas 2.1, 2.3 and 2.4.
- (ii) We establish the relationship between $\lambda_1(\alpha)$ and $\gamma_1(\alpha)$ in Lemma 2.2.
- (iii) We derive the first order differential equation for $\gamma_1(\alpha)$ by using (i) and (ii). Then by solving it, we obtain the asymptotic formula for $\lambda_1(\alpha)$.

Lemma 2.1. $\|u'_\alpha\|_2^2 = 2C_1(1 + o(1))\alpha^{(p+3)/2}\sqrt{h(\alpha)}$ for $\alpha \gg 1$.

Lemma 2.2. For $\alpha > 0$

$$(2.6) \quad \frac{d\gamma_1(\alpha)}{d\alpha} = 2\alpha\lambda_1(\alpha) - \frac{2}{p+1}\alpha^{p+1}h'(\alpha).$$

Lemma 2.3. For $\alpha \gg 1$

$$\int_0^{\|u_\alpha\|_\infty} s^{p+1}h'(s)ds = \int_I \left(\int_0^{u_\alpha(t)} s^{p+1}h'(s)ds \right) dt + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right).$$

Lemma 2.4. For $\alpha \gg 1$

$$(2.7) \quad \int_0^{\|u_\alpha\|_\infty} s^{p+1} h'(s) ds = \int_0^\alpha s^{p+1} h'(s) ds + o\left(\alpha^{(p+3)/2} \sqrt{h(\alpha)}\right).$$

Proof of Theorem 3. By simple calculation, we have

$$\frac{2}{p+1} \lambda(\alpha) \alpha^2 - \gamma(\alpha) = -\frac{p-1}{p+1} \|u'_\alpha\|_2^2 + \frac{2}{p+1} \int_I \left(\int_0^{u_\alpha(t)} s^{p+1} h'(s) ds \right) dt.$$

By this, Lemmas 2.1, 2.3 and 2.4,

$$\begin{aligned} \frac{2}{p+1} \lambda_1(\alpha) \alpha^2 - \gamma_1(\alpha) &= -\frac{2(p-1)}{p+1} C_1 \alpha^{(p+3)/2} \sqrt{h(\alpha)} (1 + o(1)) \\ &\quad + \frac{2}{p+1} \int_0^\alpha s^{p+1} h'(s) ds. \end{aligned}$$

By integration by parts,

$$\int_0^\alpha s^{p+1} h'(s) ds = \frac{1}{p+1} \alpha^{p+2} h'(\alpha) - \frac{1}{p+1} R(\alpha),$$

where

$$(2.8) \quad R(\alpha) := \int_0^\alpha s^{p+1} (h'(s) + s h''(s)) ds.$$

By this and Lemma 2.2,

$$(2.9) \quad \begin{aligned} \frac{1}{p+1} \alpha \gamma'_1(\alpha) - \gamma_1(\alpha) &= -\frac{2(p-1)}{p+1} C_1 \alpha^{(p+3)/2} \sqrt{h(\alpha)} (1 + o(1)) \\ &\quad - \frac{2}{(p+1)^2} R(\alpha). \end{aligned}$$

Now we put $\gamma_1(\alpha) = \eta(\alpha) \alpha^{p+1}$. Then for $\alpha \gg 1$, we obtain

$$(2.10) \quad \begin{aligned} \eta'(\alpha) &= -2(p-1) C_1 \alpha^{-(p+1)/2} \sqrt{h(\alpha)} (1 + o(1)) \\ &\quad - \frac{2}{p+1} R(\alpha) \alpha^{-(p+2)} \\ &:= \eta'_1(\alpha) + \eta'_2(\alpha), \end{aligned}$$

where

$$(2.11) \quad \eta_1(\alpha) = (1 + o(1)) \int_\alpha^\infty 2(p-1) C_1 s^{-(p+1)/2} \sqrt{h(s)} ds,$$

$$(2.12) \quad \eta_2(\alpha) = \frac{2}{p+1} \int_\alpha^\infty R(s) s^{-(p+2)} ds.$$

Then it is easy to show that for $\alpha \gg 1$

$$(2.13) \quad \eta_1(\alpha)\alpha^{p+1} = 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1 + o(1)).$$

We next calculate $\eta_2(\alpha)$. By (B.2), we have

$$(2.14) \quad |R(\alpha)| \leq \int_0^\alpha s^{p+1}|h'(s) + sh''(s)|ds \leq C \int_0^\alpha s^{p-\delta}ds \leq C\alpha^{p+1-\delta}.$$

By this, we easily see that $\eta_2(\alpha)$ is well defined. Then by integration by parts and simple calculation, we have

$$\begin{aligned} (2.15) \quad \eta_2(\alpha) &= \frac{2}{p+1} \int_\alpha^\infty R(s)s^{-(p+2)}ds \\ &= \frac{2}{p+1} \left[-\frac{1}{p+1} s^{-(p+1)} R(s) \right]_\alpha^\infty + \frac{2}{(p+1)^2} \int_\alpha^\infty (h'(s) + sh''(s))ds \\ &= \frac{2}{(p+1)^2} R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2} \int_\alpha^\infty (sh'(s))'ds \\ &= \frac{2}{(p+1)^2} R(\alpha)\alpha^{-(p+1)} + \frac{2}{(p+1)^2} (C_0 - \alpha h'(\alpha)). \end{aligned}$$

Therefore,

$$\begin{aligned} (2.16) \quad \gamma_1(\alpha) &= (\eta_1(\alpha) + \eta_2(\alpha))\alpha^{p+1} \\ &= 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)}(1 + o(1)) \\ &\quad + \frac{2}{(p+1)^2}(R(\alpha) + C_0\alpha^{p+1} - \alpha^{p+2}h'(\alpha)). \end{aligned}$$

By this and Lemma 2.1, we obtain

$$\begin{aligned} (2.17) \quad \frac{2}{p+1}\lambda_1(\alpha)\alpha^2 &= \gamma_1(\alpha) - \frac{p-1}{p+1}\|u'_\alpha\|_2^2 + \frac{2}{p+1} \int_0^\alpha s^{p+1}h'(s)ds \\ &\quad + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right) \\ &= 4C_1\alpha^{(p+3)/2}\sqrt{h(\alpha)} - \frac{2C_1(p-1)}{p+1}\alpha^{(p+3)/2}\sqrt{h(\alpha)} \\ &\quad + \frac{2}{(p+1)^2}C_0\alpha^{p+1} + o\left(\alpha^{(p+3)/2}\sqrt{h(\alpha)}\right). \end{aligned}$$

By this, we obtain

$$\begin{aligned} (2.18) \quad \lambda_1(\alpha) &= \frac{1}{p+1}C_0\alpha^{p-1} + (p+3)C_1\alpha^{(p-1)/2}\sqrt{h(\alpha)} \\ &\quad + o\left(\alpha^{(p-1)/2}\sqrt{h(\alpha)}\right). \end{aligned}$$

Thus the proof is complete. ■

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